Увод в теорията на операторни алгебри с приложение в математиката и физиката
Introduction to Theory of Operator Algebras with Applications to Mathematics and Physics

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Lecture 7

# Von Neumann algebras – part II

Reminder: Let $\mathcal{H}$ – Hilbert space, $\mathcal{B}(\mathcal{H}) :=$ $C^*$-algebra of all bounded operators on $\mathcal{H}$. For $\mathcal{S} \underset{set}{\subseteq} \mathcal{B}(\mathcal{H})$:

$$\mathcal{S}' := \{ A \in \mathcal{B}(\mathcal{H}) \mid AC = CA \ \forall C \in \mathcal{S} \} = \mathbb{C}[\mathcal{S}]'$$ – the commutant of the algebra generated by $\mathcal{S}$.

$\mathcal{S}'$ is a unital Banach algebra that is $C^*$-algebra if $\mathcal{S}$ is $*$-invariant ($\equiv \forall A \in \mathcal{S}\ A^* \in \mathcal{S}$).

$\mathcal{S}_1 \subseteq \mathcal{S}_2 \Rightarrow \mathcal{S}_2' \subseteq \mathcal{S}_1'$; $\mathcal{S} \subseteq \mathcal{S}'' = \mathcal{S}^{(iv)} = \dots$; $\mathcal{S}' = \mathcal{S}''' = \mathcal{S}^{(v)} = \dots$

A von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H}) \underset{def}{\Leftrightarrow} \mathcal{M}$ is $*$-invariant and $\mathcal{M}'' = \mathcal{M}$

Observation R.1 If $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ is $*$-invariant subset, $V \subseteq \mathcal{H}$ – closed, linear $P$ = orthogonal projection on $\mathcal{H}$ then:

$$V \text{ is } \mathcal{S}\text{-invariant } \left(\underset{def}{\Leftrightarrow} \forall A \in \mathcal{S} : A(V) \subseteq V\right) \Leftrightarrow P \in \mathcal{S}'.$$

Von Neumann's Theorem / Bicommutant Theorem

Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be a $C^*$-algebra which is nondegenerate ($\Leftrightarrow \overline{\mathcal{A}\mathcal{H}} = \mathcal{H}$). Then: $\mathcal{A}'' = \overline{\mathcal{A}}^w = \overline{\mathcal{A}}^s$.

Corollary $\mathcal{M}$ is von Neumann algebra $\Leftrightarrow \mathcal{M}$ is unital weakly closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$.

Theorem $\forall$ downward directed set of positive operators (="positive elements") in a von Neumann algebra $\mathcal{M}$ has an infimum in $\mathcal{M}$ and strongly converges to it.

Remark There is a theorem:

If $\mathcal{A}$ is a $C^*$-algebra such that $\forall a, b \in \mathcal{A}$ that are positive $\exists \inf(a,b)$ (i.e. $\exists c \in \mathcal{A}$ s.t. $\forall c' \in \mathcal{A} : c' \leq c \Leftrightarrow c' \leq a \ \&\ c' \leq b$) then $\mathcal{A}$ is commutative.

As an application we obtained that: $\forall A \in \mathcal{B}(\mathcal{H})$, $A^* = A$ then the commutative von Neumann algebra $\mathbb{C}[A]'' = \overline{\mathbb{C}[A]}^s$ generated by $A$ contains all spectral projections of $A$, and is uniformly generated by their linear span.

Projection (проектор) $P \in \mathfrak{M} \subseteq \mathcal{B}(\mathcal{H}) \overset{\text{def}}{\Longleftrightarrow} P^2 = P,\ P^* = P$.

$\mathcal{Z} = \mathcal{Z}(\mathfrak{M}) = \mathfrak{M} \cap \mathfrak{M}'$ – the center of the von Neumann algebra $\mathfrak{M}$
– it is a commutative von Neumann algebra ($\mathcal{Z} = \mathcal{Z}''$ – why?).

Let $P$ be a central projection in $\mathfrak{M}$ (i.e. $P \in \mathcal{Z}$, $P^2 = P$, $P^* = P$).
Then $P\mathfrak{M}P = \{PAP \mid A \in \mathfrak{M}\}$ is a weakly closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$
as well as $(1-P)\mathfrak{M}(1-P)$ and

$$\mathfrak{M} \underset{\boxed{\text{direct sum of } *\text{-algebras}}}{=} \underset{\substack{\{A|_{P\mathcal{H}} \mid A \in \mathfrak{M}\} \\ \text{a von Neumann} \\ \text{algebra on } P\mathcal{H}}}{\underbrace{P\mathfrak{M}P}_{\cong}} \oplus \underset{\substack{\{A|_{P\mathcal{H}^\perp} \mid A \in \mathfrak{M}\} \\ \text{a von Neumann} \\ \text{algebra on } (P\mathcal{H})^\perp}}{\underbrace{(1-P)\mathfrak{M}(1-P)}_{\cong}}$$

Corollary $\mathfrak{M}$ is simple (i.e. $\mathfrak{M}$ is indecompossable into a direct sum)
iff $\mathfrak{M}$ is a factor ($\overset{\text{def}}{\Longleftrightarrow} \mathcal{Z}(\mathfrak{M}) = \mathbb{C}.1$).

## 1. The lattice of projections in a von Neumann algebra

Def. A poset $(M, \leq)$ is called lattice (решетка) if $\forall x, y \in M$
$\exists \inf(x,y) =: x \wedge y$ and $\exists \sup(x,y) =: x \vee y$, where for $S \subseteq M$
$z \leq \inf S \overset{\text{def}}{\Longleftrightarrow} \forall x \in S : z \leq x$ ; $z \geq \sup S \overset{\text{def}}{\Longleftrightarrow} \forall x \in S : z \geq x$
define $\inf S$ and $\sup S$ when they exist.

If $\forall S \subseteq M\ \exists \inf S$, $\exists \sup S$ then $M$ is called complete lattice (пълна решетка)
If $\exists 0 := \inf M$, $\exists 1 := \sup M$ the lattice is called bounded.

Note If $V, W \subseteq \mathcal{H}$ – closed linear subspaces, $P, Q$ – orthogonal projections on them ($V = P\mathcal{H}$, $W = Q\mathcal{H}$) then: $V \subseteq W \Longleftrightarrow P \leq Q$.
If $P$ and $Q$ commute: $PQ = QP$ is the orthogonal projection on $V \cap W$.

Def. Ortholattice (ортрешетка) is a lattice $M$ with an operation $M \to M : x \mapsto x^\perp$, which is idempotent (идемпотентна), $x^{\perp\perp} = x$ and $x \leq y \Rightarrow y^\perp \leq x^\perp$.
Then: $(x \wedge y)^\perp = x^\perp \vee y^\perp$, $(x \vee y)^\perp = x^\perp \wedge y^\perp$.

**Remark** In a poset $(M, \leq)$, which is endowed with an operation $x \mapsto x^\perp$ s.t. $x^{\perp\perp} = x$ and $x \leq y \overset{\Longrightarrow}{(\Longleftarrow)} y^\perp \leq x^\perp$ if $\forall S \subseteq M\ \exists \inf S$ then $\forall S \subseteq M\ \exists \sup S$ and $\sup S = \left(\inf\{x^\perp \mid x \in S\}\right)^\perp$ (exercise).
Another situation in a poset when $\forall S \subseteq M\ \exists \inf S \Rightarrow \forall S \subseteq M\ \exists \sup S$ is when $M$ is bounded: $\exists \sup M$. Indeed, then:

$$\sup S = \inf\{x \in M \mid y \leq x\ \forall y \in S\} \overset{!}{\neq} \emptyset \quad \text{(exercise)}.$$

**Theorem 7.1** The poset of all projections in a von Neumann algebra $\mathfrak{M} \subseteq \mathcal{B}(\mathcal{H})$ (w.r.t. operator order) is a complete ortholattice.

**Proof.** Let $\{P_j\}_{j\in J} \subseteq \mathfrak{M} \subseteq \mathcal{B}(\mathcal{H})$ be a family of projections, $V_j := P_j\mathcal{H}$.
Let $V = \bigcap_j V_j$, $P$ - the orthogonal projection on $V$. $P_j \in \mathfrak{M} \Rightarrow V_j$ is $\mathfrak{M}'$-invariant $(\forall j \in J)$.
$\Rightarrow V$ is $\mathfrak{M}'$-invariant. $\Rightarrow P \in \mathfrak{M}'' = \mathfrak{M}$. (2 applications of Observation R1.)
Since $V \subseteq W \Leftrightarrow V^\perp \supseteq W^\perp$ the rest of the theorem follows. $\square$

**Corollary 7.2** Let $\{P_j\}_{j\in J} \subseteq \mathfrak{M}$ be an orthogonal system of projections ($\overset{\Longleftrightarrow}{\text{def}}$ $\forall j, j' \in J$, $j \neq j' \Rightarrow P_j P_{j'} = 0 = P_{j'} P_j$ ; $\forall j \in J$, $P_j \neq 0$).
Then: $\sum_j P_j = \sup\{P_j\}_{j\in J} \in \mathfrak{M}$.

**Observation 7.3.** Let $\mathfrak{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, $P \in \mathfrak{M}$ be a projection (i.e. $P^2 = P$, $P^* = P$) and $V = P\mathcal{H}$. Then if $W = \overline{\mathfrak{M}\cdot V}$ and $Q$ is the orthogonal projection on $W$ then $Q \in \mathcal{Z}(\mathfrak{M})$.
**Proof.** Note that $\mathfrak{M}\cdot V$ is $\mathfrak{M}$-invariant and $\mathfrak{M}'$-invariant (since if $C \in \mathfrak{M}'$ then for $\forall A \in \mathfrak{M}$, $v \in V$: $CAv = CAPv = APCv \in \mathfrak{M}\cdot V$). Then $W = \overline{\mathfrak{M}\cdot V}$ is also $\mathfrak{M}$ and $\mathfrak{M}'$-ivariant. $\Rightarrow Q \in \mathfrak{M}' \cap \mathfrak{M}'' = \mathcal{Z}(\mathfrak{M})$. $\square$

**Exercise.** (we shall not use it) Under the conditions of Observation 7.1:

$$W = \bigcap\{P'\mathcal{H} \mid P' \in \mathfrak{M}' \text{ is a projection}, P \leq P'\}$$

$$= \bigcap\{Q'\mathcal{H} \mid Q' \in \mathcal{Z} \text{ is a projection}, P \leq Q'\}$$

**Hint.** Let $P_1$ and $Q_1$ be the orthogonal projections on the right hand sides of line 1 and line 2, respectively. Then $P_1 \in \mathfrak{M}'$, $Q_1 \in \mathcal{Z}$.

**Def.** The constructed projection $Q \in \mathcal{Z}$ in Observation 7.1 is called central support of the projection $P \in \mathfrak{M}$.

## 2. Partial isometries (частични изометрии).

**Def.** Let $\mathcal{H}$ be a Hilbert space, $U : \mathcal{H} \to \mathcal{H}$ is called partial isometry iff $U_0 := U|_{(\operatorname{Ker} U)^\perp} : (\ker U)^\perp \xrightarrow{\sim} U\mathcal{H}$ (= Image of $U$) is an isometry. (In particular, $U \in \mathcal{B}(\mathcal{H})$.) Denote $V := (\operatorname{Ker} U)^\perp$, $W = U\mathcal{H}$, then:

$$\begin{array}{ccccc} V & \oplus & V^\perp & = & \mathcal{H} \\ {\scriptstyle \text{isometry } U_0} \downarrow \wr & & \downarrow 0 & & \downarrow U \\ W & \oplus & W^\perp & = & \mathcal{H} \end{array} \qquad \begin{array}{ccccc} V & \oplus & V^\perp & = & \mathcal{H} \\ {\scriptstyle \text{isometry } U_0^{-1} = U_0^*} \uparrow \wr & & \uparrow 0 & & \uparrow U^* \\ W & \oplus & W^\perp & = & \mathcal{H} \end{array} \qquad \text{(diagrams)}$$

Hence, $U^*U =: P$ - the orthogonal projection on $V$,
$UU^* =: Q$ - the orthogonal projection on $W$.

$V$ is called the initial space of $U$, $W$ - target space of $U$.

Notation: $V \underset{U}{\xrightarrow{\sim}} W$ and also $P \underset{U}{\xrightarrow{\sim}} Q$.

<u>Observation 7.4</u> The following conditions are equivalent:
(1) $U \in \mathcal{B}(\mathcal{H})$ is a partial isometry;
(2) $U^*U$ is a projection;
(3) $UU^*$ is a projection;
(4) $U^*UU^* = U$;
(5) $UU^*U = U^*$.

<u>Proof.</u> (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (3), (1) $\Rightarrow$ (4), (1) $\Rightarrow$ (5) follow by (diagrams).
(2) $\Rightarrow$ (3): Set $Q = UU^* = Q^*$. Then $Q^3 = U(U^*U)(U^*U)U^* = UU^*UU^* = Q^2$.
$\Rightarrow \sigma_{\mathcal{B}(\mathcal{H})}(Q) \subseteq \{0,1\}$ since $(1-\lambda Q)^{-1} = \sum_{n=0}^{\infty} \lambda^n Q^n = 1 + \lambda Q + \frac{\lambda^2}{1-\lambda} Q^2$
$(Q^n = Q^{n-1} = \dots = Q^2)$. Then $Q$ is a projection.
(3) $\Rightarrow$ (2): replacing $U = U^{**} \leftrightarrow U^*$. (4) $\Rightarrow$ (2): by $\underline{U|}$. (5) $\Rightarrow$ (3): by $\underline{|U^*}$.
Finally: (2) & (3) $\Rightarrow$ $P \underset{U}{\xrightarrow{\sim}} Q$ i.e. (1) (one checks (diagrams)). $\square$

<u>Note:</u> (4) $\Rightarrow U^*QU = P$. (5) $\Rightarrow UPU^* = Q$.

## 3. Comparison of projections in a von Neumann algebra

**Def.** Let $\mathcal{M}$ be a von Neumann algebra in $\mathcal{B}(\mathcal{H})$. $P, Q \in \mathcal{M}$ - projections

$P \sim Q \underset{\text{def}}{\Longleftrightarrow} \exists U \in \mathcal{M}$ - partial isometry $P \underset{U}{\xrightarrow{\sim}} Q$

$\underset{\text{by Observation 7.4}}{\Longleftrightarrow} \exists U \in \mathcal{M}$ s.t. $P = U^*U$ and $Q = UU^*$.

For the corresponding closed linear subspaces $V = P\mathcal{H}$, $W = Q\mathcal{H}$ we shall also write $V \sim W$.
(But note that possibly not $\forall V \subseteq \mathcal{H}$-closed linear, is of a form $V = P\mathcal{H}$ for $P \in \mathcal{M}$-projection.)

Observations 7.5 (a) Let $\tilde{u} \in \mathcal{M}$ be unitary ($\tilde{u}\tilde{u}^* = 1 = \tilde{u}^*\tilde{u}$). $P \in \mathcal{M}$-projection.
Then: $P \sim \tilde{u} P \tilde{u}^*$ (i.e. for $V = P\mathcal{H}$: $V \sim \tilde{u} V$).
Proof. Set: $u = \tilde{u} P$. Then $u^* u = P \tilde{u}^* \tilde{u} P = P^2 = P$, $u u^* = \tilde{u} P^2 \tilde{u}^* = \tilde{u} P \tilde{u}^*$. □

(b) $\sim$ is equivalence relation.
Proof. 1) $P \sim 1^* P 1 = P$. 2) If $P \xrightarrow[u]{\sim} Q$ then $Q \xrightarrow[u^*]{\sim} P$.
3) If $P_1 \xrightarrow[u_1]{\sim} P_2$, $P_2 \xrightarrow[u_2]{\sim} P_3$ then $P_1 \xrightarrow[u_2 u_1]{\sim} P_3$, $u_1, u_2 \in \mathcal{M}$. □

Remark. If $P \sim Q$ it is not necessary that $Q = \tilde{u} P \tilde{u}^*$ for a unitary $\tilde{u} \in \mathcal{M}$.
Also: $P \sim Q \not\Rightarrow 1 - P \sim 1 - Q$ (i.e. $V \sim W \not\Rightarrow V^\perp \sim W^\perp$).

Def. Let $P, Q \in \mathcal{M}$ - projections.
$$P \precsim Q \underset{def}{\Longleftrightarrow} \exists \text{ projection } P_1 \in \mathcal{M}, \; P \sim P_1 \leq Q.$$
We shall prove bellow that the relation $\precsim$ induces a partial order on the equivalence classes $[P]$ of projections in $\mathcal{M}$.

## 4. Polar decomposition in a von Neumann algebra

Let $A \in \mathcal{B}(\mathcal{H})$. We know that $\exists!$ positive operator $:= |A| \in \overline{\mathbb{C}[A^*A]}$ s.t. $A^*A = |A|^2$.
Note $|A| \neq |A^*| = (AA^*)^{1/2}$, in general.

Theorem 7.6. $\exists$ partial isometry $u \in \mathbb{C}[A, A^*]''$ s.t. $A = u|A|$.

Lemma $\forall A \in \mathcal{B}(\mathcal{H})$: $\overline{A\mathcal{H}} = (\operatorname{Ker} A^*)^\perp$
Proof. $u \perp \overline{A\mathcal{H}} \Leftrightarrow \langle u, Av \rangle = 0 \; (\forall v \in \mathcal{H}) \Leftrightarrow \langle v, A^* u \rangle = 0 \; (\forall v \in \mathcal{H}) \Leftrightarrow A^* u = 0$. □

Corollary 7.7 $\overline{|A|\mathcal{H}} = (\operatorname{Ker} |A|)^\perp = (\operatorname{Ker} A)^\perp = \overline{A^*\mathcal{H}}$.
Proof: $Av = 0 \Leftrightarrow 0 = \|Av\|^2 = \langle v, A^*Av \rangle \Leftrightarrow 0 = \langle v, |A|^2 v \rangle = \||A|v\|^2 \Leftrightarrow |A| v = 0$. □

Proof of Thm 7.6 Let $V_0 = |A|\mathcal{H}$, $V = \overline{V_0}$. Set: $u_0 : |A| v \mapsto Av \; (\forall v \in \mathcal{H})$.
Then $\langle u_0 |A| v, u_0 |A| w \rangle = \langle Av, Aw \rangle = \langle v, A^*Aw \rangle = \langle |A|v, |A|w \rangle$.
$\Rightarrow u_0$ defines an unitary embedding $\overline{V_0} \rightarrowtail \mathcal{H}$. Set $u|_{V_0^\perp} = 0$
$u|_{\overline{V_0}} = u_0$ – this defines a partial isometry. Then $A = u|A|$ by the construction.

Why $\mathcal{U} \in \mathbb{C}[A, A^*]''$?

Let $P$ be the orthogonal projection on $\overline{V_0}$. Then $P$ is projection on $\ker |A^*|$ (see corollary 7.7) - spectral projection. $\Rightarrow P \in \mathbb{C}[A^*A]''$. If $C \in \mathbb{C}[A, A^*]'$ then $C\mathcal{U}|A|v = CAv = ACv = \mathcal{U}|A|Cv = \mathcal{U}C|A|v$ so that $C\mathcal{U} = \mathcal{U}C$ on $\overline{V_0}$.
On $V_0^\perp$: $C\mathcal{U}|_{V_0^\perp} = 0$, $\mathcal{U}C|_{V_0^\perp} = \mathcal{U}C(1-P)|_{V_0^\perp} = \mathcal{U}(1-P)C|_{V_0^\perp} = 0$.
$\Rightarrow C\mathcal{U} = \mathcal{U}C$. $\Rightarrow \mathcal{U} \in \mathbb{C}[A, A^*]''$ $\square$

<u>Remark</u> We used the fact that: if $B \in \mathcal{B}(\mathcal{H})$, $B = B^*$ then the orthogonal projection on $\operatorname{Ker} B$ is the spectral projection $\Pi$ corresponding to $\{0\} \subseteq \mathbb{R}$.
Indeed: $Bv = 0 \Leftrightarrow Bf(B)v = 0 \;\; \forall f \in C(\mathbb{R})$

$$\Leftrightarrow f(B)v = 0 \;\; \forall f \in C^\infty(\mathbb{R}) \text{ s.t. } f(0) = 0 \Leftrightarrow (1 - \Pi)v = 0.$$

<u>Corollary 7.8 (a)</u> Let $\mathfrak{M}$ be a von Neumann algebra in $\mathcal{B}(\mathcal{H})$, $A \in \mathfrak{M}$, $P$ be the orthogonal projection on $\overline{A\mathcal{H}}$ = $\overline{\text{Image of } A}$ and $Q$ be the orthogonal projection on $\overline{A^*\mathcal{H}}$ = $\overline{\text{Image of } A^*}$. Then $P, Q \in \mathfrak{M}$ and $P \sim Q$.

<u>Proof.</u> $P\mathcal{H} = \ker |A^*|$, $Q\mathcal{H} = \operatorname{Ker} |A|$ - spectral projections.
$\Rightarrow P, Q \in \mathbb{C}[A, A^*]'' \subseteq \mathfrak{M}$. By Corollary 7.7 $Q\mathcal{H} = \overline{|A|\mathcal{H}}$. Then:
$\overline{A^*\mathcal{H}} = \overline{|A|\mathcal{H}} \xrightarrow[\mathcal{U}]{\sim} \overline{A\mathcal{H}}$ by the polar decomposition $A = \mathcal{U}|A|$ with $\mathcal{U} \in \mathbb{C}[A, A^*]'' \subseteq \mathfrak{M}$. $\square$

<u>Remark.</u> $\mathcal{U}$ in Thm 7.6 is unique if it is a partial isometry $\overline{A^*\mathcal{H}} \xrightarrow[\mathcal{U}]{\sim} \overline{A\mathcal{H}}$.
(since $\mathcal{U}(|A|v) = Av$.)

<u>Remark.</u> The projection $P$ on $\overline{A\mathcal{H}}$ is called left support of $A \in \mathfrak{M}$.
$P = \inf \{ P_1 \in \mathfrak{M} \mid P_1 A = A \}$. The right support of $A$ is defined similarly:
$Q = \inf \{ Q_1 \in \mathfrak{M} \mid A Q_1 = A \}$. By using the polar decomposition one proves also that $P \sim Q$.

<u>Corollary 7.8 (b)</u> Let $P, Q \in \mathfrak{M}$ be projections, $V = P\mathcal{H}$, $W = Q\mathcal{H}$. Then $V \ominus (V \cap W^\perp) \sim W \ominus (W \cap V^\perp)$ (i.e. the corresponding projections are equivalent: $P - (P \wedge (1-Q)) \sim Q - (Q \wedge (1-P))$).

<u>Proof.</u> $\overline{QP\mathcal{H}} = \overline{QV} = V \ominus (V \cap W^\perp)$, $\overline{PQ\mathcal{H}} = \overline{PW} = W \ominus (W \cap V^\perp)$.
But $(PQ)^* = QP$. $\square$

<u>Note:</u> $V \ominus (V \cap W^\perp) = 0 \Leftrightarrow V = V \cap W^\perp \Leftrightarrow V \subseteq W^\perp \Leftrightarrow V \perp W$.

<u>Corollary 7.8 (c)</u> Let $P_1, P_2 \in \mathcal{M}$ be projections and $\nexists P_1', P_2' \in \mathcal{M}$ – projections s.t. $P_1' \neq 0 \neq P_2'$, $P_1' \leq P_1$, $P_2' \leq P_2$ and $P_1 \sim P_2$. Then if $Q_k$ is the central support of $P_k$ $(k=1,2)$, i.e. $Q_k \in \mathcal{Z}(\mathcal{M})$ is the orthogonal projection on $\overline{\mathcal{M} P_k \mathcal{H}}$ (see Observation 7.2) it follows that $Q_1 Q_2 = 0$.

In other words, $\exists F \in \mathcal{Z}$ - projection s.t. $P_1 \leq F$ and $P_2 \leq 1-F$.

<u>Proof.</u> Let $V_k = P_k \mathcal{H}$ $(k=1,2)$. $\forall U_1, U_2 \in \mathcal{M}$ – unitary: $U_1 V_1 \perp U_2 V_2$, otherwise

$$0 \neq \underbrace{U_1 V_1 \ominus (U_1 V_1 \cap (U_2 V_2)^{\perp})} \sim \underbrace{U_2 V_2 \ominus (U_2 V_2 \cap (U_1 V_1)^{\perp})} \neq 0.$$

$$0 \neq V_1' \sim U_1 V_1' = U_1 (V_1 \ominus V_1'') \sim U_2 (V_2 \ominus V_2'') = U_2 V_2' \sim V_2' \neq 0$$

– contradiction $(V_1' \subseteq V_1,\ V_2' \subseteq V_2)$

Thus, $\underbrace{\mathrm{Span}\{U V_1 \mid U \in \mathcal{M} \text{ - unitary}\}}_{\mathcal{M} V_1} \perp \underbrace{\mathrm{Span}\{U V_2 \mid U \in \mathcal{M} \text{ - unitary}\}}_{\mathcal{M} V_2}$.

due to the following:

<u>Lemma</u> $\forall$ unital $C^*$-algebra $\mathcal{A}$: $\mathcal{A} = \mathrm{Span}\{U \mid U \in \mathcal{A} \text{ is unitary}\}$.

<u>Proof.</u> $\forall a \in \mathcal{A}$: $a = \lambda_1 c_1 + \lambda_2 c_2$, $c_k = c_k^*$, $\|c_k\| < 1$ (for instance: $a = 2\|a\| \frac{a+a^*}{4\|a\|} + i\, 2\|a\| \frac{a-a^*}{4i\|a\|}$). Then $c_k = \frac{1}{2}\underbrace{\left(c_k + i(1-c_k)^{1/2}\right)} + \frac{1}{2}\underbrace{\left(c_k - i(1-c_k)^{1/2}\right)}$ unitary

## 5. Comparison theorems

Let $P, Q \in \mathcal{M}$ - projections in a von Neumann algebra

<u>Theorem A</u> If $P \precsim Q$ and $Q \precsim P$ then $P \sim Q$.

<u>Theorem B</u> $\exists F \in \mathcal{Z}(\mathcal{M})$ - projection s.t. $FP \precsim FQ$, $(1-F)Q \precsim (1-F)P$.

<u>Corollaries 7.9 (a)</u> Let $[P]$ be the equivalence class of a projection $P \in \mathcal{M}$ w.r.t. $\sim$. Then $[P] \leq [Q] \overset{\text{def}}{\Longleftrightarrow} P \precsim Q$ defines a poset structure on $\{[P] \mid P \in \mathcal{M}\text{-projection}\}$.

(b) If $\mathcal{M}$ is a factor then $\{[P] \mid P \in \mathcal{M}\text{-projection}\}$ is linearly ordered, i.e. either $[P] \leq [Q]$ or $[Q] \leq [P]$ $\forall P, Q \in \mathcal{M}$ - projections.

<u>Lemma 7.10.</u> Let $\{P_j\}_{j \in J} \subseteq \mathcal{M}$, $\{Q_j\}_{j \in J} \subseteq \mathcal{M}$ be two orthogonal families of projections s.t. $\forall j \in J$: $P_j \sim Q_j$. Then $\sum_j P_j \sim \sum_j Q_j$.

<u>Proof.</u> Let $P_j \overset{\sim}{\underset{u_j}{\longrightarrow}} Q_j$, $u_j \in \mathfrak{M}$ – partial isometry. Then for $\forall S \underset{\text{finite}}{\subseteq} J$ set $u_S := \sum_{j \in S} u_j$. Hence, $\{u_S\}_{S \underset{\text{finite}}{\subseteq} J}$ – a net. Then $u_S \xrightarrow{s} u =: \sum_j u_j$ and $P \overset{\sim}{\underset{u}{\longrightarrow}} Q$, $u \in \mathfrak{M}$. □

<u>Proof of Theorem A</u> We have $P =: P_0 \overset{\sim}{\underset{u}{\longrightarrow}} Q_1 \leq Q$ and $Q =: Q_0 \overset{\sim}{\underset{u'}{\longrightarrow}} P_1 \leq P$.

We construct then $P_0 \geq P_1 \geq P_2 \geq P_3 \geq \cdots$, $Q_0 \geq Q_1 \geq Q_2 \geq Q_3 \geq \cdots$

$$Q_{k+1} := u P_k u^* \geq 0, \quad P_{k+1} := u' Q_k u'^* \geq 0.$$

Since:
$$u(P_{2k} - P_{2k+1})u^* = Q_{2k+1} - Q_{2k+2} \geq 0$$
$$u'(Q_{2k} - Q_{2k+1})u'^* = P_{2k+1} - P_{2k+2} \geq 0$$

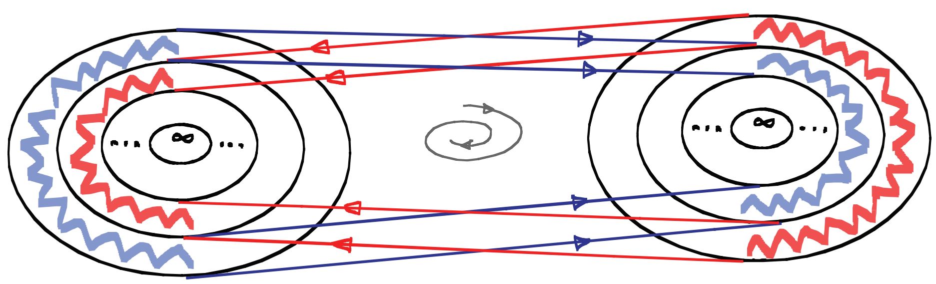
by induction: since the l.h.s. are positive

$\Rightarrow$ $P_0 \geq P_1 \geq P_2 \geq P_3 \geq \cdots$, $Q_0 \geq Q_1 \geq Q_2 \geq Q_3 \geq \cdots$

Set $P_\infty = \inf P_k$, $Q_\infty = \inf Q_k$. $\Rightarrow$ $u P_\infty u^* = Q_\infty$, $P_\infty \overset{\sim}{\underset{uP_\infty}{\longrightarrow}} Q_\infty$.

then:
$$\begin{array}{ccccccc}
P = & P_\infty & + & \sum_{k=0}^{\infty} (P_{2k} - P_{2k+1}) & + & \sum_{k=0}^{\infty} (P_{2k+1} - P_{2k+2}) & \\
 & uP_\infty \big\downarrow \wr & & u(P_{2k} - P_{2k+1}) \big\downarrow \wr & & u'(Q_{2k} - Q_{2k+1}) \big\uparrow \wr & \\
\sim & Q_\infty & + & \sum_{k=0}^{\infty} (Q_{2k+1} - Q_{2k}) & + & \sum_{k=0}^{\infty} (Q_{2k} - Q_{2k+1}) & = Q.
\end{array}$$
□

<u>Remark:</u>



The same scheme applies in set theory

<u>Proof of Theorem B.</u> Consider the poset consisting of pairs $(\{P_j\}_{j\in J}, \{Q_j\}_{j\in J})$ of orthogonal families of projections $\{P_j\}, \{Q_j\} \subseteq \mathcal{M}$ s.t. $\forall j$ $P_j \sim Q_j$, $P_j \leq P$, $Q_j \leq Q$. The partial order is: $(\{P_j\}_{j\in J}, \{Q_j\}_{j\in J}) \leq (\{P'_j\}_{j\in J'}, \{Q'_j\}_{j\in J'})$ iff $\{P_j\} \subseteq \{P'_{j'}\}$, $\{Q_j\} \subseteq \{Q'_{j'}\}$. It follows by the Zorn lemma (check!) that there are maximal elements in this poset. Set $\mathcal{M} \ni P_0 := \sum_j P_j \leq P$, $\mathcal{M} \ni Q_0 := \sum_j Q_j \leq Q$ for such maximal families $\{P_j\}$ and $\{Q_j\}$.

Let $P_1 = P - P_0$, $Q_1 = Q - Q_0$. By maximality $\nexists\, P_1', Q_1' \in \mathcal{M}$-projections s.t. $P_1' \neq 0 \neq Q_1'$, $P_1' \leq P_1$, $Q_1' \leq Q_2$ and $P_1' \sim Q_1'$. By Corollary 7.8 (c) $\exists F \in Z(\mathcal{M})$-projection s.t. $P_1 = P - P_0 \leq 1 - F$, $Q_1 = Q - Q_0 \leq F$. Then $(P - P_0)F = 0$ and $PF = P_0 F \sim Q_0 F \leq QF$, where $P_0 \xrightarrow[U]{\sim} Q_0 \Rightarrow P_0 F \xrightarrow[UF]{\sim} Q_0 F$ since $F \in Z$ $\left( (UF)^* UF = U^* U F^2 = PF,\ UF(UF)^* = \cdots = QF \right)$.

Similarly, $(Q - Q_0)(1 - F) = 0. \Rightarrow Q(1-F) = Q_0(1-F) \sim P_0(1-F) \leq P(1-F)$. $\square$

<u>Remark:</u> Scheme of the proof